

Projective and Affine Spatial Operators for Regions

ROLAND BILLEN

Unit of Geomatics

University of Liege

17 Allée du 6-Août, 4000 Liege

BELGIUM

ELISEO CLEMENTINI

Dept. of Electrical Engineering

University of L'Aquila

I-67040 Poggio di Roio (AQ)

ITALY

Abstract: - Among Geographic Information Systems functionality is the retrieval of spatial data by using various spatial operators. The proposal of this paper is about the definition of new spatial operators that are based on projective and affine geometric invariants. Such a special category of geometric properties, despite being not widely used in previous work, reveals itself a very interesting realm for the exploration of new operators that are both powerful and easy to use from an end-user perspective.

Key-Words: - Projective space, Affine space, Invariants, Spatial operators, Geographic information systems.

1 Introduction

Spatial operators in Geographic information systems (GIS) are used in queries against spatial data. Current standards are those provided by the OpenGIS Consortium (OGC) [6] and implemented in software by main GIS and database systems vendors. OGC spatial operators include topological operators that has been first described in [2, 5]. Topological properties are able to describe primitive aspects of geometry and for this reason they have been used for the development of operators that are suitable for end-user interaction. As it is claimed in [3], the study of other geometric properties (projective and affine) will give us the opportunity of proposing other qualitative operators in favor of GIS interaction. Projective and affine properties are not as general as topological ones but can be investigated without the power of metric spaces.

In this paper, we develop projective and affine operators, moving from the study of basic invariants of projective and affine spaces. Basic projective invariants are collinearity and cross ratio. Basic affine invariants are parallelism and distance ratio. The work is subdivided in two parts: first, we introduce unary operators, then we focus on binary operators. Unary operators extract information about a single object's geometry (size and shape), independently from its context. Binary operators extract information about the relation between two objects. Relations between more than two objects can be inferred by re-applying binary operators.

2 Unary operators

In this section, we discuss all the unary projective and affine operators. Some of them (the *order 0*, the *convex hull* and the *inflection point*) are basic operators that

should be generally used in combination with others. Then, depending on the kind of qualitative information that can be extracted from their use, three groups of operators can be identified: the first group is (*number of order 0*, *straight line*); the second group is (*number of inflection points*, *number of concavities*, *inner visibility* and *outer visibility*); the third group is (*barycentre*).

2.1 Order 0 operator

The order 0 operator is related to the convexity properties of the order of points of a closed convex set. This operator is mainly inspired from the concepts of the *dimensional model* [1] which proposes an affine space portioning of objects. This operator extracts the points of region's boundary that have order 0. Qualitatively, a point of a region's boundary has order 0 if it corresponds to a broken part of a boundary's line. A region may have several order 0 points (e.g. the square), or a single order 0 point (e.g. the drop) or no order 0 points (e.g. the disk) (see Fig. 1).

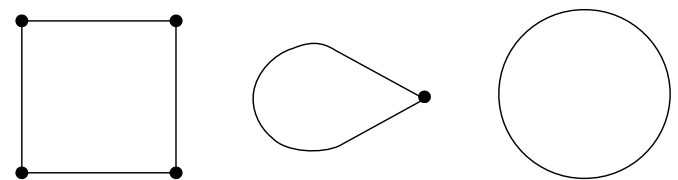


Fig.1 Examples of region without and with order 0 points.

The 0 order points give us very useful information when they are combined with other operators and relations (see Section 3.2), or when they are combined with a *number of component* operator (see Section 2.4).

2.2 Convex Hull operator

The convex hull operator returns the convex hull of an object. As the 0 order operator, the convex hull operator is very useful when it is combined with other operators or relations.

2.3 Inflection point operator

Inflection points are an important characteristics of a region's boundary. They are needed in some other operators (*type of concavities* and *inner visibility*). An inflection point of a curve is change in curvature from concave to convex or conversely.

2.4 Number of Order 0 points operator

The number of components of order 0 in an object is a way to distinguish among spatial objects. If a region has no order 0 points, it can be called **smooth**. If a region has between 1 and 3 order 0 components, it can be called **slightly sharp**. If a region has more than 3 order 0 components, it can be called **sharp** (see Fig. 2).

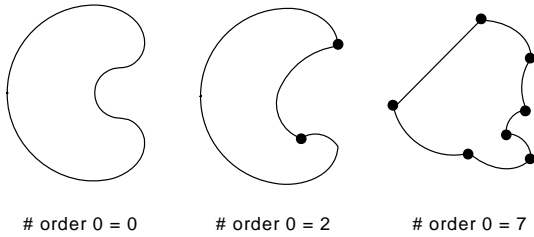


Fig. 2. Distinctions among regions, based on the number of order 0 points.

2.5 Straight line operator

The projective property of collinearity can be used in combination with the order 0 operator. It is interesting to look at the curvature of the pieces of boundary between the order 0 points. The straight line operator returns true if the piece of boundary bounded by two successive order 0 points is a straight segment. If all the boundary segment between order 0 points are straight, then the region is called a **polytope** (it is possible to refine this interpretation when the object is a convex set, by counting the number of order 0 points : triangle, quadrilateral, etc.). If more than half boundary segments between order 0 points are straight, then the region can be called **mostly straight**. If less than half boundary segments between order 0 points are straight, but still some are straight, then the region is called **partly straight** (see Fig. 3).

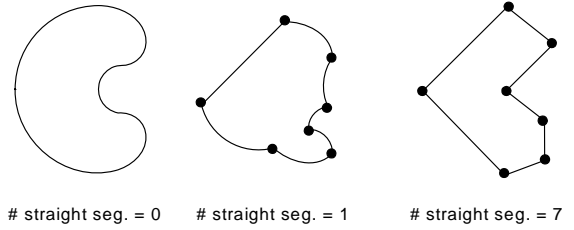


Fig. 3. Distinctions among regions, based on the number of straight segments.

2.6 Number of inflection points operator

This operator counts the number of inflection points. Whereas it could be used alone, it is especially interesting when using it in conjunction with the *type of concavities* operator (see Section 2.8).

2.7 Number of concavities operator

The number of concavities operators results from the combination of the convex hull operator, and the number of components of a region operator. The number of concavities of a region is, therefore, the number of components of the set difference between the convex hull of the object and the object itself (see Fig. 4).

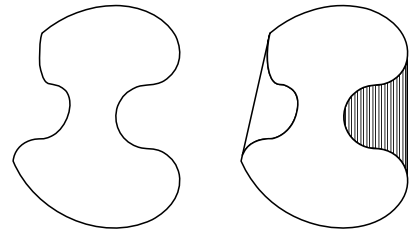


Fig. 4. A region with two concavities.

2.8 Type of concavity operator

The type of concavities is based on the number of inflection points. Specifically, a simple concavity is a concavity that contains only two inflection points. A complex concavity is a concavity which has more than two inflection points. We do not develop here a deeper description. See also Cohn [4] for related work on number and types of concavities.

2.9 Inner visibility operator

As a consequence of the collinearity invariant, a concept of visibility can be defined in a projective space. A point of a region is visible from another one if the linear segment between them is entirely contained in the region. The *visibility region* can be defined as the set of all the points from which the whole region is visible. In a convex region, all the points of the region are visible from any other point of the region: in this case, the visibility region corresponds to the region itself.

For non convex regions, it is possible to determine the visibility region with a geometric construction using inflection points of the region's boundary. We limit

ourselves to simple concavity cases. The operation consists of the determination of the intersection (called **visibility point**) of the tangents that pass through the two inflection points and their respective intersections with the boundary of the object (see Fig. 5). First, consider the cases where the two tangents meet the boundary only once. If the visibility point lays inside the region, the visibility region is the region bounded by the two tangents from the visibility points to the intersections with the region's boundary and the portion of region's boundary between such two points.

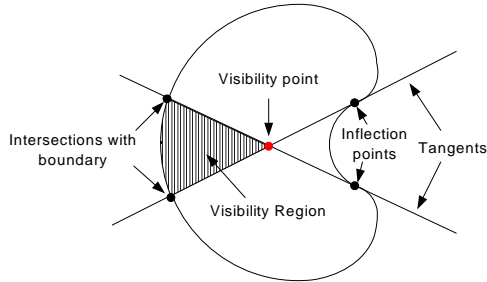


Fig. 5. Construction of the visibility region.

If the visibility point lays outside the object, there is no visibility region. Other different configurations can be distinguished, by considering the position of the visibility point in relation to inflection points, intersections with the boundary and the convex hull (see Fig. 6-7).

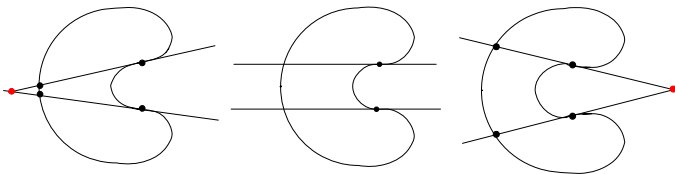


Fig. 6. Different configurations based on the position of the visibility point.

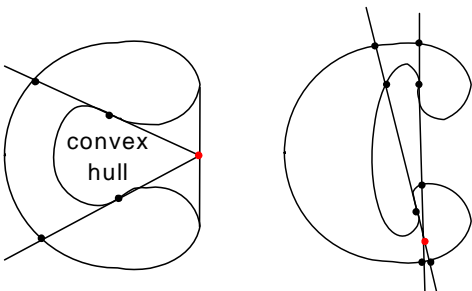


Fig. 7. Different configurations based on the position of the visibility point with respect to the convex hull.

We can propose the following qualitative classification of regions with a simple concavity based on inner visibility operator. **Slightly curved**: when the tangents meet once the boundary and when the visibility point is inside the region. **Medium curved**: when the visibility point is outside the region, the tangents meet once the

boundary and the intersections with boundary are between the visibility point and the inflection points (with the limit case where the tangents are parallel).

Strongly curved: when the visibility point is outside the region, the tangents meet once the boundary and the inflection points are between the visibility point and the meeting points. **Very strongly curved**: when the visibility is outside the region but inside the convex hull or when at least one of the tangents meets three times the boundary.

In the case of straight edges, all that have been discussed is still valid, with the exception of the determination of the inflection points. In some cases, the two inflection points and the visibility point can be superimposed (see Fig.8).

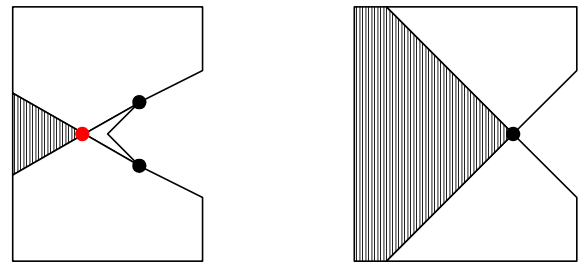


Fig. 8. The determination of the visibility region for straight boundaries.

2.10 Outer visibility operator

The concept of visibility can be applied directly to concavities, resulting in other interesting classifications of regions' shape. If the tangents meet more than one time the boundary or if the visibility points is outside the object but inside the convex hull, we use the **outer visibility operator**. This operator allows us to determine the part of the boundary of the region that is not visible from the outside of the convex hull of the region. A point of the boundary of the concavity is said **shaded** if there is no line passing through it and meeting only twice the boundary (see Fig. 9). The shaded part of the boundary may have several components. We can talk in that case of **1-shaded** or **2-shaded** concavity. If we have more than 2 components, the concavity must be complex. The shaded part of the boundary and the shaded part of the convex hull of the object form the **shaded region**.

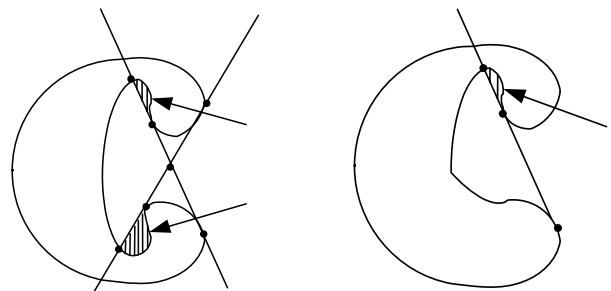


Fig. 9. A concavity with 2-shaded components and 1-shaded component, respectively.

2.11 Barycentre operator

The barycentre (or gravity centre) is an affine invariant. When a region is not convex (and therefore has concavities), it is interesting to check whether the barycentre of the region lays inside the region or outside. For a non convex region, if the barycentre is inside the region, the region can be called *superficially curved*. If the barycentre is outside the region, the region can be called *deeply curved* (see Fig. 10).

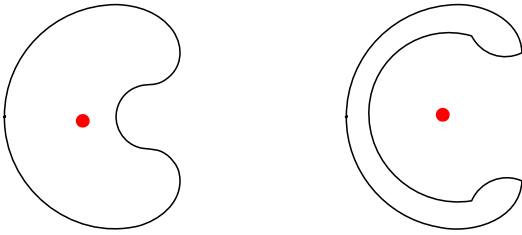


Fig. 10. Classification of regions based on the position of the barycentre.

3 Binary operators

The binary operators hereby presented can be classified in two categories. The first category can be seen as a refinement of topological operators, applying topological relations not only between original regions but also between derived region's parts, such as barycentre, convex hull, order 0 points, visible region and shaded region. The second category contains operators that are not related with topological relations or their refinements.

3.1 Topological relations using derived affine and projective parts

Topological operators can be applied to derived affine and projective parts, due to the fact that these derived geometric objects are still object of the OGC spatial data model: for example, the application of the order0 operator to a region returns in general a set of points (belonging to the spatial type *multipoint* in the OGC data model). Let us consider the configurations in Fig. 11. They correspond to equivalent topological relations between regions A and B. The same configurations are actually distinct, if affine operators are used. By using the CBM operators [2], the topological relation of all the three cases is A overlap B. A distinction can be made between the first case and the other two cases by using the order0 operator: the first case is (A, order0) disjoint B, while the other two cases are (A, order0) touch B. The distinction between the second case and third case

can be done by counting the number of intersection components between (A, order0) and B, which are 1 and 2 respectively.

Similarly, it is possible to use other derived parts, such as the convex hull, the visibility region and the shaded region, and check topological relations among these parts. For instance, A region A can therefore be qualified as *visible from* an object B if object B is inside region A's visibility region. An object B is *shaded by* a region A, if B is inside the shaded region of A (see Fig. 12).

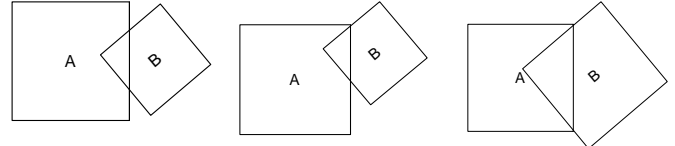


Fig. 11. Configurations that can be distinguished by combining affine operators with topological operators.

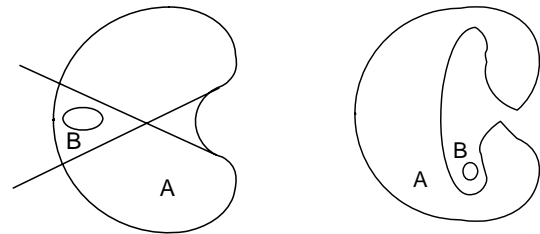


Fig. 12. These configurations can be expressed by A is visible from B, and B is shaded by A, respectively.

3.4 Affine direction and affine distance

The two basic invariants in affine space are the parallelism and the distance ratio. It is possible to use these two invariants to express the relative distance and orientation of two objects, without making use of metric concepts. The approach is based on the application of a grid with an arbitrary cell dimension. In an affine space, either the cell grid is a parallelogram, a square, a rectangle or a lozenge, the only constraint is that the opposite edges of the cell grid must remain parallel. Hence, under affine transformations, the grid cell may vary from parallelogram to rectangle or from square to lozenge.

The arbitrary choices are the definition of the grid cell and its principal direction. When these are fixed, it is possible to establish a relative distance and orientation between two objects, which are invariants under affine transformations. Let's take the case of an ellipse and a circle (Fig. 13). Different solutions for the choice of the grid are possible. A first solution is to use as a grid cell a parallelogram that can be built on the main axes of the ellipse. We can see in this case that the circle is at a distance of three grid cells away from the ellipse, whereas its direction is East. As a second solution for the grid, if we took the Minimum Bounding Rectangle

(MBR) of the ellipse as a grid cell, the circle would be at a distance of two grid cells and still to the East direction. It is very interesting to note that such relative distances and directions are not necessary symmetric. In fact, if we take the MBR of the circle as the grid cell, we can see that the ellipse is at a distance of three/four grid cells from the circle. The asymmetry in relative distances may well correspond to how differently distances are perceived from a small object or a big object point of view. Smaller objects appear relatively closer from the bigger object's point of view, and vice versa: e.g., from a Sun reference system, the Earth would seem closer, while from a Earth reference system the Sun would seem farther (whenever their absolute distance in the Solar system is of course the same).

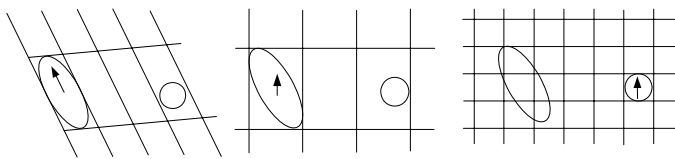


Fig. 13. Different grids to establish the affine distance between a circle and an ellipse.

The proposed affine system for relative distance and direction operators can also be useful to introduce further distinctions in the case of the relation between a region with a concavity and another region. In Fig. 14, the three configurations are topological equivalent; furthermore in all cases the round region is contained inside the concavity of the curved region. It is possible to distinguish among them by using the relative distance and direction system. In the first case, the smallest object is relatively *far* from the biggest one (even if it is included in his convex hull) where in the second case, it is *very close* in at least one direction. In the third case, the two object are *close*.

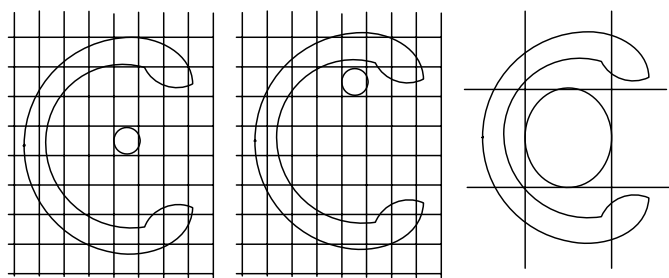


Fig. 14. Different configurations by affine relative distance.

4 Conclusions

Spatial operators are the basic tools to retrieve and analyse spatial data. The power of a spatial database engine (the core of a GIS) can be defined on the number and the applicability of the spatial operators that are

being offered. After decades of great variability of spatial operators that were included in various systems, a 'de facto' standard has been reached by the spatial data types and operators of the OGC, which has been implemented by major GIS vendors.

In the authors' opinion, the set of OGC operators is too restrictive, being limited to topological operators and a few basic metric operators. Increased functionality would be gained by the introduction of other qualitative operators. In this paper, projective and affine properties have been investigated and new projective and affine operators have been proposed. It has been shown that the new operators can be used in conjunction with existing topological operators to refine the geometric configurations that can be distinguished. Furthermore, a concept of affine relative distance is useful to represent in qualitative terms the relative position between two spatial objects.

Further work will be about the formal definitions of the operators, which have been informally introduced in this paper. Also, we plan to implement these operators and show by means of experiments their applicability towards end-user interaction.

Acknowledgements:

This paper has been supported by M.U.R.S.T. under project "Rappresentazione ed elaborazione di dati spaziali nei Sistemi Informativi Territoriali".

References:

1. Billen, R., et al., *The dimensional model: A framework to distinguish spatial relationships*, in *Advances in Spatial Data Handling*. 2002, Springer-Verlag. p. 285-298.
2. Clementini, E. and P. Di Felice, *A Model for Representing Topological Relationships Between Complex Geometric Features in Spatial Databases*. Information Sciences, 1996. **90**(1-4): p. 121-136.
3. Clementini, E. and P. Di Felice, *Spatial Operators*. ACM SIGMOD Record, 2000. **29**(3): p. 31-38.
4. Cohn, A.G., *A hierarchical representation of qualitative shape based on connection and convexity*, in *Spatial Information Theory: A Theoretical Basis for GIS - International Conference, COSIT'95*, A.U. Frank and W. Kuhn, Editors. 1995, Springer-Verlag: Berlin. p. 311-326.
5. Egenhofer, M.J. and J.R. Herring, *Categorizing Binary Topological Relationships Between Regions, Lines, and Points in Geographic Databases*. 1991, Department of Surveying Engineering, University of Maine, Orono, ME.
6. OpenGIS Consortium, *OpenGIS Simple Features Specification for SQL*. 1998.